EXTENDED SUPER-KP HIERARCHIES AND GENERALIZED STRING EQUATIONS

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Abstract

We propose a consistently algebraic formulation of the extended KP (supersymmetric) integrable-hierarchy systems. We exploit the results already established in [14] and which consist in a framework suspected to unify in a fascinating way all the possible supersymmetric KP-hierarchies and then their underlying supergravity theories . This construction leads among other to built explicit non standard integrable Lax evolution equations suspected to reduce to the well known KP integrable equation. We present also a contribution of our construction to the subject of string equation and solitons. Other algebraic properties are also presented.

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1 Introduction

An interesting subject which have been studied recently from different point of views deals with the field of non linear integrable systems and their higher and lower spin extensions [1, 2]. These are exactly solvable models exhibiting a very rich structure in lower dimensions and are involved in many areas of mathematical physics. One recall for instance the two dimensional Toda(Liouville) fields theories [3, 2] and the KdV and KP hierarchies models [4, 5], both in the bosonic as well as in the supersymmetric case.

Non linear integrable models, are associated to systems of non linear differential equations which can be solved exactly. Mathematically these models have become more fascinating by introducing some new concepts such as the infinite dimensional Lie (super) algebras.[6], Kac-Moody algebras [7], W-algebras [1, 2], quantum groups [8] and the theory of formal pseudodifferential operators [4, 5]. From the physical point of view only recently after the discovery of the connection between the generalized KdV hierarchies and the matrix models formulation of two dimensional gravity [9], has there been a great progress in the study of these systems. Wile the bosonic non linear integrable models have been studied quite extensively, not much is known, in general, about the supersymmetric extension.

The most widely studied supersymmetric integrable models which have many interesting properties is the super-KdV (sKdV) system [10]. Note by the way that a super-integrable model which exhibits a very rich structure and which in fact leads to the sKdV system upon reduction is the supersymmetric KP equation of Manin and Radul [11]. Another interesting integrable system which have been also generalized recently to the supersymmetric case is the two boson system [12]. This model known also as the dispersive long water wave equation has a very rich structure since it is :1- Tri Hamiltonian , 2- has a non standard Lax representation and 3- Reduces to well known integrable systems under appropriate reduction.

Non linear integrable models are also known to connect with the recent subject of string theory and its various applications [9]. This deep connection is established through solitons which characterise in some sense non linear physical models [13]. For a review, recall that in the late of 60-ies and early 70-ies a large group of physicists developed the very beautiful theory of the bosonic quantum strings. They used in this issue standard operator quantisation, decomposing fields in the Fourier series and replacing c-numbers by operators with standard canonical commutators. This program was effectively realized for the ?zero-loop? described by the Riemann surfaces of the zero genus. The underlying Virasoro algebra (conformal symmetry) and its representations are known to play an important role. This program stopped because nobody was able to quantize fields in such a way for non zero genus. In early 80-ies Polyakov solved the problem of quantization of bosonic strings using the functional path integral. Later on many physicist focussed to complete much more this program by using the analytical construction of the Soliton theory in Riemann surfaces. Recently a modern terminology was introduced to

describe the connection between the theory of solitons based on pseudo-differential operators and string theory. This consist in what we know as ?string equation? which means exactly the following pseudo-differential equation:

$$[L, A] = 1 \tag{1}$$

String equation appeared in 1989-90 years after the pioneering works of Gross, Migdal, Brezin, Kazakov, Douglas, Shanker, David and many others in matrix models formulation of gravity theories [9]. In this work ; we propose a consistently algebraic formulation of the extended KP integrable hierarchy systems both in the bosonic and supersymmetric cases. We exploit the results already established in [14] and which consist in a framework suspected to unify in a fascinating way all the possible supersymmetric KP-hierarchies and then their underlying supergravity theories . This construction leads among other to built explicit non standard integrable Lax evolution equations suspected to reduce to the well known KP integrable equation. We present also a contribution of our construction to the subject of string equation. Other algebraic properties are also presented.

2 The Lie algebra of pseudo differential operators

In this section, we give the general setting of the basic properties of the tensor algebra of the bosonic W currents of conformal spin $s > 1$. This is a semi-infinite dimensional space of the infinite tensor algebra of arbitrary integer spin fields. Recall that W currents, one of the main actors in two dimensional conformal theory and their higher spin extensions, are analytic fields obeying a nonlinear closed algebra. The W fields also appear in the study of higher differential operators involved in the analysis of nonlinear integrable models to be considered in this article. We shall start however by defining our notations.

The two dimensional Euclidean space $R^2 \cong C$ is parametrized by $z = t + ix$ and $\bar{z} = t - ix$. As a matter of convention, we set $z = z^+$ and $z = z^-$ so that the derivatives $\partial/\partial z$ and $\partial/\partial \bar{z}$ are, respectively, represented by $\partial_+ = \partial$ and $\partial_- = \bar{\partial}$. The so(2) Lorentz representation fields are described by one component tensors of the form $\psi_k(z, \bar{z})$ with $2k \in Z$. Z is the set of relative integers. In two dimensional conformal field theories (CFT), an interesting class of fields is given by the set of analytic fields $\phi_k(z)$. These are $SO(2) \cong U(1)$ tensor fields that obey the analyticity condition $\partial_- \phi_k(z) = 0$. In this case the conformal spin k coincides with the conformal dimension. Note that under a $U(1)$ global transformation of parameter θ, the object $z^{\pm}$, $\partial_{\pm}$ and $\phi_k(z)$ transform as

$$z^{\pm\prime} = e^{\mp i\theta} z^{\pm}, \quad \partial'_{\pm} = e^{\pm i\theta} \partial_{\pm}, \quad \phi'_k(z) = e^{ik\theta} \phi_k(z) \tag{2}$$

so that $dz\partial_z$ and $(dz)^k \phi_k(z)$ remain invariant. In a pure bosonic theory, which is the purpose of the present study, only integer values of conformal spin k are involved. We denote by $\Xi^{(0,0)}$ the tensor algebra of analytic fields of arbitrary conformal spin. This is a completely reducible

infinite dimensional SO(2) Lorentz representation (module) that can be written as

$$\Xi^{(0,0)} = \oplus_{k \in Z} \Xi_k^{(0,0)} \tag{3}$$

where the $\Xi_k^{(0,0)}$'s are one dimensional $SO(2)$ spin k irreducible modules. The upper indices $(0,0)$ carried by the spaces figuring in Eq.(3) are special values of general indices (p,q) to be introduced later on.

Next we introduce the space of pseudo-differential operators whose elements $L_s^{(p,q)}$ are the generalization of the well known differential Lax operators involved in the analysis of the so called KdV-hierarchy and in Liouville (Toda) field theories. The simplest example is given by the Hill operator

$$L_2 = \partial^2 + u_2(z) \tag{4}$$

which play an important role in the study of Liouville field theory and in the KdV equation. A natural generalization of the above relation is given by [15]

$$L_s = \sum_{i=p}^{q} u_{s-i}(z)\partial^i \tag{5}$$

where $u_{s-i}(z)$ s are analytic fields of spin $(s-i)$ and where p and q with $p \leq q$ are integers that we suppose positive for the moment. We shall refer hereafter to p as the lowest degree of $L_s^{(p,q)}$ and q as the highest degree. We consider these two features of Eq (5) by setting

$$\deg(L_s^{(p,q)}) = (p,q) \tag{6}$$

and

$$\Delta(L_s^{(p,q)}) = s \tag{7}$$

as been the conformal weight. Note that the KdV operator Eq(2.3) is discovered from Eq(2.4) as a special case by setting $s = 2$, $p = 0$ and $q = 2$ together with the special choice $u_0 = 1$ and $u_1 = 0$ corresponding to special choice dealing with Lie algebraic $sl(2)$. Moreover, Eq(5) which is well defined for $0 \leq p \leq q$, may be extended to negative integers by introducing pseudo-differential operators of type $\partial^{-i}, i \geq 1$ whose action on the fields $u_s(z)$ is defined as

$$\partial^{-k} u_s(z) = \sum_{l=0}^{\infty} (-1)^l c_{k+l-1}^l u_s^{(l)} \partial^{-k-l} \tag{8}$$

We have

$$\partial^k \partial^{-k} u_s(z) = u_s(z) \tag{9}$$

As it was previously noted, a natural representation basis of nonlinear pseudodifferential operators of spin m and negative degrees (p,q) is given by

$$\delta_m^{(p,q)}(u) = \sum_{i=p}^{q} u_{m-i}(z)\partial^i \tag{10}$$

This configuration is useful in the study of the algebraic structure of the spaces $\Xi_m^{(p,q)}$ and $\Xi^{(p,q)}$. Note also that we can use another representation of pseudodifferential operators, namely, the Volterra representation. This is convenient in the derivation of the second Hamiltonian structure of higher conformal spin integrable theories. Moreover, ne sees that operators with negative lowest degrees p and positive highest degrees q denoted by $D_m^{(p,q)}[u]$ split as

$$D_m^{(p,q)}[u] = \delta_m^{(p,q)}(u) + d_m^{(p,q)}(u) \tag{11}$$

More generally we have

$$D_m^{(p,q)}[u] = D_m^{(p,k)}(u) + D_m^{(k+1,q)}(u) \tag{12}$$

for any integers $p \leq k \leq q$. As a consequence, one finds that

$$(p,q) = (p,k) + (k+1,q) \tag{13}$$

for any three integers such that $p \leq k < q$. Now let $\Xi_m^{(p,q)}$; m, p, and q integers with $q \geq p$, be the set of spin m differential operators of degrees (p,q), the $\Xi_m^{(i,i)}$'s are one dimensional spaces given by

$$\Xi_m^{(i,i)} = \Xi_{m-i}^{(0,0)} \otimes \partial^i \tag{14}$$

3 Unified framework of bosonic KP-hierarchy

Recall that KP hierarchies can be thought of as a dynamical system defined on a space whose functions $u_j(z)$ are elements of the ring of analytic fields. It is also defined as the universal family of isospectral deformations of the pseudo-differential operator;

$$L = \partial + \sum_{i=1}^{\infty} u_i(z)\partial^{1-i} \tag{15}$$

satisfying the evolution equation

$$\frac{\partial L}{\partial t_n} = \left[(L^n)_+, L\right], n = 1, 2, ... \tag{16}$$

where the subscripts $+$ means taking purely differential part of (L^n). Now; we are very interested in generalizing this standard integrable KP-hierarchy. A natural way to do it is to focus the conformal spin decomposition. This allows to suppose the existence of three classes of integrable KP-hierarchies generalizing the standard ones and which are labelled by the conformal spin quantum number $s = \pm, 0$. A part of the Lax operator Eq() generating Σ_+ one can introduce two other classes of integrable hierarchies which are described by the following Lax operators

$$L_0 = u_{-1}\partial + \sum_{i=0}^{\infty} u_i(z)\partial^{-i} \tag{17}$$

generating the subspace $\sum_0$ of Lorentz scalar pseudo-operators and

$$L_{-} = u_{-2}\partial + \sum_{i=-1}^{\infty} u_i(z)\partial^{-i-1} \tag{18}$$

which belongs to Σ_{-}. Actually, we now well known that the origin of these three classes of integrable KP-hierarchy is traced to the fact that there exist precisely three decompositions of Σ into a linear sum of two subspaces namely [14-15]

$$\Sigma = \oplus_{s=\pm,0}\left(\Sigma_s^{+} \oplus \Sigma_s^{-}\right) \tag{19}$$

as given in Eq(). This provide then a unified framework in which all the possible KP-hierarchies described by various Lax operators can be written.

4 Unified framework of supersymmetric KP-hierarchy

4.1 The space of higher spin supersymmetric Lax operators

The aim of this section is to describe the supersymmetric version of the space of bosonic Lax operators introduced previously. This supersymmetric generalization which is straightforward and natural in the fist steps, exhibits however, some properties which are ignored in the bosonic case and make the fermionic study more fruitful. Using the space of supersymmetric Lax operators, one can derive the Hamiltonian structure of non linear two dimensional super integrable models extending the bosonic Hamiltonians.

Let us first consider the ring of all analytic super fields $u_{\frac{k}{2}}(\hat{z}),\ k \in Z$, which depend on $(1 \mid 1)$ superspace coordinates $\hat{z} = (z, \theta)$. In this super commutative Z_2-graded ring R, one can define an odd super derivation $D = \partial_\theta + \theta\partial$, the N=1 supercovariant derivatives which obeys the N=1 supersymmetric algebra $D^2 = \partial$ with $\theta^2 = 0$ and $\partial_\theta = \int d\theta$.

Note that the supersymmetric G.D bracket, which we shall discuss in the sequel, defines a Poisson bracket on the space of functional of the superfields $u_{\frac{k}{2}}(\hat{z})$ defined on the ring $R\left[u(\hat{z})\right]$.

We define the ring $\Sigma\left[D\right]$ of differential supersymmetric operators as polynomials in D with coefficients in R. using our previous notation, one set

$$\Sigma[D] = \oplus_{m\in Z}\oplus_{p\leq q}\Sigma_{\frac{m}{2}}^{(p,q)}[D], p, q \in Z \tag{20}$$

where $\Sigma_{\frac{m}{2}}^{(p,q)}[D]$ is the space of supersymmetric operators type

$$\pounds_{\frac{m}{2}}^{(p,q)}[u] = \sum_{i=p}^{q} u_{\frac{m-i}{2}}(\hat{z})D^i \quad p, q \in Z \tag{21}$$

$\Sigma_{\frac{m}{2}}^{(p,q)}$ behaves as a $(1 + q - p)$ dimensional superspace. Note also that the ring R of all graded superfields can be decomposed as

$$R \equiv R^{(0,0)} := \oplus_{k\in Z} R_{\frac{k}{2}}^{(0,0)} \tag{22}$$

where $R^{(0,0)}_{\frac{k}{2}}$ is the set of superfield $u_{\frac{k}{2}}(\hat{z})$ indexed by half integer conformal spin $\frac{k}{2} \in Z+\frac{1}{2}$. Thus, the one dimensional objects $u_{\frac{m-i}{2}}(\hat{z})D^i$ are typical elements of the superspace

$$\Sigma^{(i,i)}_{\frac{m}{2}} = R^{(0,0)}_{\frac{m-i}{2}}.D^i \equiv R^{(0,0)}_{\frac{m-i}{2}} \otimes \Sigma^{(i,i)}_{\frac{i}{2}} \tag{23}$$

which is fundamental in the construction of supersymmetric operators type (3.2).

The expression Eq(3.4) means also that

$$\Sigma^{(p,q)}_{\frac{m}{2}}[D] \equiv \oplus^{q}_{i=p} \Sigma^{(i,i)}_{\frac{m-i}{2}} \tag{24}$$

Indeed these are important in the sense that one easly identify all objects of the huge superspace Σ.

An element $\pounds$ of $\Sigma[D]$ is called a supersymmetric Lax operator if it is homogeneous under the Z_2-grading

$$|x| := \begin{cases} 0 & , x \ \ even \\ 1 & , x \ \ odd \end{cases} \tag{25}$$

And have the following form at order $n, n \in N$

$$\pounds^{(0,n)}_{\frac{n}{2}} := \sum_{i=0}^{n} u_{\frac{i}{2}}(\hat{z}) D^{n-i} \tag{26}$$

The homogeneity condition simply states that the Z_2-grading of the N=1 superfield $u(\hat{z})$ is defined by

$$\left| U_{\frac{i}{2}}(\hat{z}) \right| = i \, (mod\, 2) \tag{27}$$

The space of supersymmetric Lax operators is refereed hereafter to as $\Sigma^{(0,n)}_{\frac{n}{2}}$ and exhibits a dimension n+1.

1. We recall that the upstairs integers (0,n) are the lowest and the highest degrees of $\pounds$ and the down stair index $\frac{n}{2}$ is the spin of $\pounds$. To define a Lie algebraic structure on the superspace Σ one need to introduce a graded commutator defined for two arbitrary operators X and Y by

$$[X, Y]_i = XY - (-)^i YX \tag{28}$$

Where the index i=0 or 1 refer to the commutator [,] or anticommutator {,} respectively. The multiplication of operators in Σ is defined be the super Leibniz rule given by the following mapping

$$D^{(l)} : R^{(0,0)}_{\frac{j}{2}} \longrightarrow \Sigma^{(p,l)}_{\frac{j+l}{2}}$$

$$D^{(l)} \phi\,(\hat{z}) = \sum_{i=0}^{\infty} \begin{bmatrix} l \\ l-i \end{bmatrix} (-)^{j(l-i)} \phi^{(i)} D^{(l-i)} \tag{29}$$

Where l is an arbitrary integer and the super binomial coefficients $\begin{bmatrix} l \\ k \end{bmatrix}$ are defined by [..]

$$\begin{bmatrix} l \\ k \end{bmatrix} = \begin{cases} 0, fork \succ landfor(k,l) = (0,1) mod 2 \\ \begin{pmatrix} l \\ k \end{pmatrix}, otherwise \end{cases} \tag{30}$$

The lowest degree p of the superspace $\Sigma^{(p,l)}_{\frac{j+l}{2}}$ Eq(3.12) is given by

$$p = \begin{cases} 0, if l \geq 0 \\ -\infty if l \leq -1 \end{cases} \tag{31}$$

The symbol [x] stands for the integer part of x$\in Z/2$ and $\begin{pmatrix} i \\ j \end{pmatrix}$ is the usual binomial coefficient.

4.1.1 The 5=2x2+1 splitting:

Recall that there are two usual supersymmetric extensions of the standard KP-hierarchy Eq (3.1). The first one is given by the Manin-Radul supersymmetric KP hierarchy associated with the odd super Lax operator[11]:

1.

$$L_{MR} = D + \sum_{i\geq 1}^{\infty} u_{\frac{i+1}{2}}(\hat{z})D^{-i} \tag{32}$$

The second one is given by the supersymmetric KP-hierarchy associated with the Figueroa-Mass- Ramos even super Lax operator [16]

$$L_{FMR} = D^2 + \sum_{i\geq 0}^{\infty} u_{\frac{i+1}{2}}(\hat{z})D^{-i} \tag{33}$$

An important consequence of the choice of L_{FMR}, Eq(4.16), is that under suitable choice of reduction, it reduce to the Inami-Kanno super Lax operator describing the generalised N=2 super KdV-hierarchy [17]. Recall also the important remark of [14], in which a unifying framework of the previous well known super KP-hierarchies is proposed. Indeed, using the $6 = 3 \times 2$ decomposition of the Lie superalgebra Ξ, Eq(3.6), we have shown that there exist precisely $5 = 2 \times 2 + 1$ classes of supersymmetric KP-hierarchies. The origin of these hierarchies is traced to the fundamental fact that there exist precisely five grading algebras:

$$\begin{aligned} g_1 &= \Xi^{+}_{+,\bar{0}} \oplus \Xi^{-}_{+,\bar{0}} \\ g_2 &= \Xi^{+}_{+,\bar{1}} \oplus \Xi^{-}_{+,\bar{1}} \\ g_0 &= \Xi^{+}_{0,\bar{0}} \oplus \Xi^{-}_{0,\bar{0}} \\ g_1^* &= \Xi^{+}_{-,\bar{1}} \oplus \Xi^{-}_{-,\bar{1}} \\ g_2^* &= \Xi^{+}_{-,\bar{0}} \oplus \Xi^{-}_{-,\bar{0}} \end{aligned}$$

5 References

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